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Bounds on the Hausdorff Measure of the Minkowski Sausage

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Abstract. We produce a formula to calculate increasingly tight lower bounds on the Hausdorff measure of the Minkowski sausage. We also present a non-trivial construction for an upper limit on the Minkowski sausage, then state some conjectures and directions for future work.

1. Introduction

The Minkowski sausage, also known as the type-2 quadratic Koch curve, is a well-known example of a self-similar fractal with a Hausdorff Dimension equal to 1.5. It has been used as an example in computing applications and in physical simulations, such as in [5][7] and [1]. While the Hausdorff measure of some fractals, such as the Sierpinski gasket, have been studied, the Hausdorff measure of most fractals is considered in general to be a difficult problem [8]. Specifically, the Hausdorff measure of the Minkowski sausage remains uncalculated. Using the techniques in [3], it is straightforward but computationally expensive to calculate upper bounds on fractals satisfying the open set condition. Furthermore, lower bounds for the Koch Curve were obtained using a modification [4]. While this method does not numerically outperform other methods that have been developed for the Sierpinski gasket [6], it is of interest for its ability to approach the measure indefinitely with an exponentially decreasing error.

Theorem 1. *The Hausdorff Measure of the Minkowski sausage has the following upper bound:*

$$H^s(M) \le 0.6094265648090738$$

Theorem 2. *The Minkowski sausage satisfies the following lower bound:*

$$H^s(M) \ge a_n e^{-36\sqrt{5}(\frac{1}{4})^n}$$

where a_n is defined below and calculable, in theory, to any level of accuracy.

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Let $D \subset \mathbb{R}^n$ be a nonempty set. $E \subset \mathbb{R}^n$ is a self-similar set defined by m similar contracting maps $S_i : D \to D$, with contracting ratios, $0 < c_i < 1 (i = 1, 2, ..., m)$. Let E satisfy the *Open Set Condition* (*OSC*), that is, that there exists a nonempty open set U such that $S_i(U) \cap S_j(U) = \emptyset$ for $i \neq j$, and $U \supseteq S_i(U)$ for all i. Then

$$dim_H(E) = s$$

$$0 < H^s(E) < \infty$$

Where s satisfies $\sum_{i=1}^{m} c_i^s = 1$, $dim_H(E)$ and $H^s(E)$ denote the Hausdorff dimension and measure of E, respectively [2]. Let $J_n = \{(i_1 i_2 ... i_n) : 1 \leq i_1, i_2, ... i_n \leq m\}$ and $E_{i_1 i_2 ... i_n} = S_{i_1} \circ S_{i_2} \circ ... \circ S_{i_n}(E)$, which is self-similar to E. Then we have $E = \bigcup_{J_n} E_{i_1 i_2 ... i_n}$. Throughout the paper, the diameter of any set A will be represented as $|A|$.

Proposition 1.1 and 1.2 are taken from [3], and restated here for completeness.

Proposition 1.1. *Suppose that E is a self-similar set satisfying the open set condition. For* $n \geq 1$, $1 \leq k \leq m^n$, *let* $\Delta_1, \Delta_2, ..., \Delta_k \in \{E_{i_1 i_2 ... i_n} : 1 \leq i_1, i_2, ... i_n \leq m\}$ *and* μ *be the common self-similar probability measure on E,* $\mu(E_{i_1 i_2 ... i_n}) = c_{i_1}^s c_{i_2}^s ... c_{i_n}^s$. *Let*

$$b_k = \min_{\substack{\Delta_i \in E_{i_1 i_2 ... i_n} \\ i=1,2,...,k}} \left\{ \frac{\left| \bigcup_{i=1}^{k} \Delta_i \right|^s}{\mu\left(\bigcup_{i=1}^{k} \Delta_i \right)} \right\}$$

where the minimum is taken over all possible unions of k choices of $E_{i_1 i_2 ... i_n}$. *Let* $a_n = \min_{1 \leq k \leq m^n}\{b_k\}$. *If there exists a constant* $A > 0$ *such that* $a_n \geq A (n = 1, 2, ...)$, *then* $H^s(E) > A$.

We will say that a set $A = \bigcup_{i=1}^{k} \Delta_i$ *satisfies* a_n if $\frac{|A|^s}{\mu(A)} = a_n$. We will also sometimes use Δ^n or Δ_i^n to specify that the Δ_i are from level n of the construction of E.

Proposition 1.2. *As n increases,* a_n *decreases, and* $\lim_{n \to \infty} a_n = H^s(E)$.

It is noted that if $c_1 = c_2 = ... = c_m = c$, then in the definition of b_k, $\mu\left(\bigcup_{i=1}^{k} \Delta_i \right)$ may be replaced with kc^{ns}.

1.1. The Minkowski sausage

The Minkowski sausage, named after Herman Minkowski, has been used as another example of a basic self-similar fractal akin to the Sierpinski gasket or Koch curve. To construct it, Let $D = M_0$ be the convex quadrilateral with points $(0, 0), (\frac{1}{3}, \frac{1}{3}), (1, 0), (\frac{2}{3}, -\frac{1}{3})$. let M be the unique self-similar attractor of the function system $\{S_1, S_2, ..., S_8\}$, with

$S_1(M) = \frac{1}{4}M$

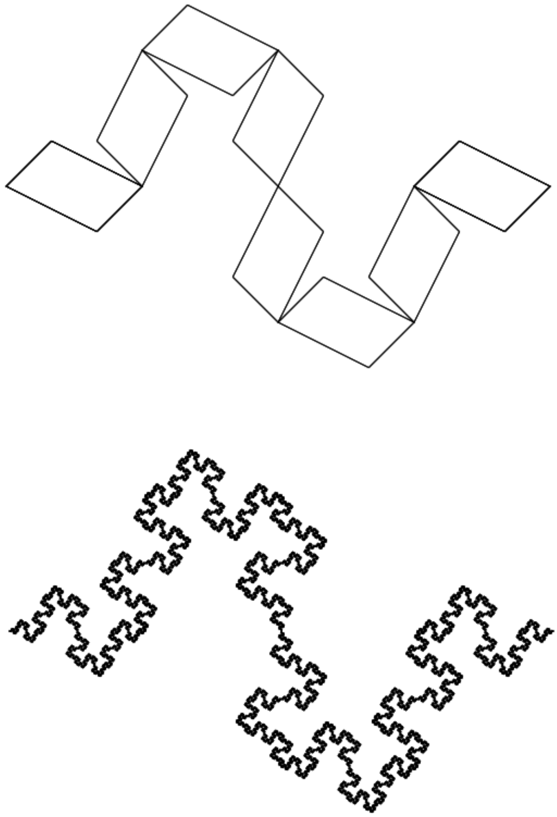

Figure 1. The first steps of the construction of the Minkowski sausage, $n = 1$ and $n = 5$

$$S_2(M) = \tfrac{1}{4}[R(M, \tfrac{\pi}{2})] + (\tfrac{1}{4}, 0)$$
$$S_3(M) = \tfrac{1}{4}M + (\tfrac{1}{4}, \tfrac{1}{4})$$
$$S_4(M) = \tfrac{1}{4}[R(M, -\tfrac{\pi}{2})] + (\tfrac{1}{2}, \tfrac{1}{4})$$
$$S_5(M) = \tfrac{1}{4}[R(M, -\tfrac{\pi}{2})] + (\tfrac{1}{2}, 0)$$
$$S_6(M) = \tfrac{1}{4}M + (\tfrac{1}{2}, -\tfrac{1}{4})$$
$$S_7(M) = \tfrac{1}{4}[R(M, \tfrac{\pi}{2})] + (\tfrac{3}{4}, -\tfrac{1}{4})$$
$$S_8(M) = \tfrac{1}{4}M + (\tfrac{3}{4}, 0)$$

Where $R(M, \theta)$ is a counterclockwise rotation of M by θ radians, and the addition and scaling are point-wise. The resulting fraction has a contraction ratio c of $\frac{1}{4}$, and with 8 copies created in each level of construction, let $m = 8$. Thus, the Hausdorff dimension is $s = \log_4 8 = 1.5$.

2. Upper Bound

The following construction develops the ideas of [6], namely, that a good candidate for a_n can be found using circle intersections. The main observation is that just as the convex hull of a shape has the same diameter as the original shape, one may extend the sides of the shape to be arcs of a circle without increasing the diameter. Thus, it is sensible that any K satisfying a_n would be interior of some set of intersecting circles. The following construction is not optimized — the exact values were found via human-mediated inspection after an initial guess — but it is also a non-trivial upper bound.

Let $B(p, r)$ be the ball with radius r centered at the point p. The following lemma, while not showing how to find a_n, restricts the possibilities inherent in any set which does satisfy a_n. It gives a way to test whether a set which purports to be a close estimation of a_n can be easily improved or not, and formalizes the above notion of using circle intersections.

Proposition 2.1. *Let $E \subset \mathbb{R}^n$ be a self-similar fractal satisfying the open set condition, and $K = \bigcup_i \Delta_i^n$ be a union of $\Delta_i^n \in \{E_{i_1 i_2 \ldots i_n}\}$. Let $O = \bigcap_{j \in J} B(p_j, |K|)$ be an intersection of balls, where $\{p_j\}$ is some set of points indexed by the set $j \in J$. Then K satisfies a_n only if there exists O such that $K = \bigcup_j \left\{\Delta_j^n : \Delta_j^n \subset O\right\}$. Further, if the convex hull of K is a polytope, an O exists such that J is finite.*

Proof. Let $\bar{K}$ be the convex hull of K. It is a classical result that $|K| = |\bar{K}|$. For any set of points A, let $\odot_A$ be the following circle intersections:

$$\odot_A = \bigcap_j \{B(p_j, |A|) : p_j \in A\}$$

Take any point $p \in \odot_{\bar{K}}$. Then $|K \cup p| = |K|$, because otherwise, $\mathrm{dist}(p, q) > |K|$ for some point q in K, so $q \notin B(q, |K|)$, and $p \notin \odot_{\bar{K}}$. So $|\bar{K} \cup p| = |K|$.

Notice that although $\odot_{\bar{K} \cup p} \subseteq \odot_{\bar{K}}$, any point $q \in \odot_{\bar{K}}$ within a distance of $|K|$ from p is not excluded. We also know that $|\Delta_i| \leq |K|$ for any Δ_i, since K is the union of potentially several Δ_i. Thus, if $p \in \Delta_i$ for some Δ_i, and $\Delta_i \subset \odot_{\bar{K}}$, then $|K| = |K \cup \Delta_i|$. So $\frac{|K \cup \Delta_i|^s}{(k+1)c^{ns}} < \frac{|K|^s}{kc^{ns}}$, and by definition, K cannot satisfy a_n. So either $K = \bigcup_j \{\Delta_j^n : \Delta_j^n \subset \odot_{\bar{K}}\}$, which is in the form we wanted, or K does not satisfy a_n.

The finiteness condition of the number of intersecting balls is clear after noticing that if $\bar{K}$ is a polytope, then letting V be the set of vertices of $\bar{K}$, that $\odot_V = \odot_{\bar{K}}$ (let p be in $\bar{K}$: since $\bar{K}$ is the convex hull, p is a linear combination of the vertices of $\bar{K}$, and $B(p, |K|)$ follows).

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Theorem 2.1.

$$H^s(M) \leq 0.6094265648090738$$

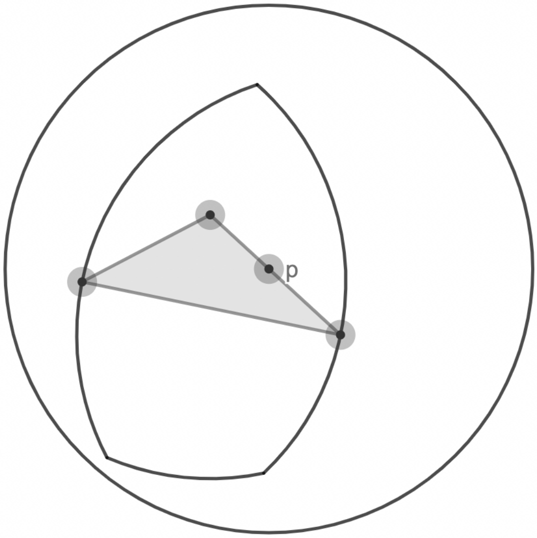

Figure 2. An example where K is a triangle. The outer circle is $B(p,|K|)$, and the curve of intersecting arcs is $\odot_K$. Notice that $\odot_K \subset B(p,|K|)$.

Proof. In the spirit of proposition 2.1, let $r_0 = \frac{\sqrt{312218}}{3072} \approx 0.1818895537$, and define Θ to be the intersections of

$$\begin{aligned}\Theta = &B\Big(\Big(\frac{59}{128}, \frac{69}{256}\Big), r_0\Big)\\ &\cap B\Big(\Big(\frac{1679}{3072}, \frac{335}{3072}\Big), r_0\Big)\\ &\cap B\Big(\Big(\frac{7}{12}, \frac{1}{6}\Big), r_0\Big)\end{aligned}$$

Let $\Theta_n = \bigcup_{i=1}^{k} \left\{\Delta_i^n : \Delta_i^n \subset \Theta\right\}, 1 \le k \le m$. In other words, Θ_n is the set of all Δ^n which fit entirely within the circle intersections Θ. Once we have constructed Θ_n, let $\theta_n = \frac{|\Theta_n|^s}{kc^{ns}}$ (In this way, $H^s(M) \le a_n \le \theta_n$ for any n). Θ_n is constructed in such a way that it is easy to compute the values in table 1.

Table 1. Values of θ_n

n	k	$\lvert\Theta_n\rvert$	θ_n
2	5	0.1767766952966369	0.951365692002177
3	60	0.17921510973005467	0.6474118500200924
4	515	0.18175609568956003	0.6162915263508869
5	4171	0.18188955370398896	0.6094265648090738

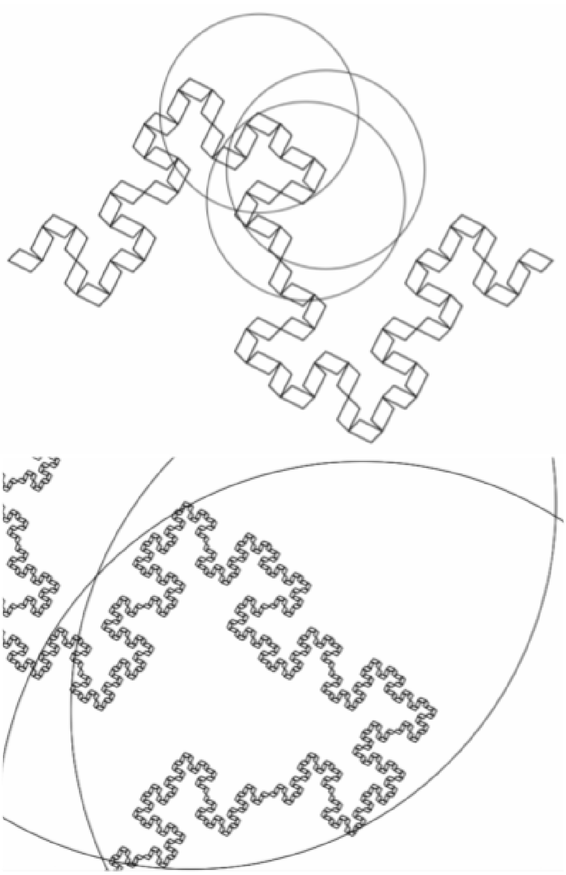

Figure 3. Θ_2 and Θ_5.

For computational reasons, to find θ_n, we use maps of the constructor M_0, not E itself. For a shape such as the Sierpinski gasket, this posed no issue because the constructor, an equilateral triangle, has as the same convex hull as Sierpinski gasket itself. In our construction, it should be noted that M_0 may result in larger diameters than using M itself. Thus, our values for θ_n are strictly larger than could be achieved by using a more accurate constructor. Since $M \subset M_0$, the inequality is valid and

$$H^s(M) \leq a_5 \leq \theta_5 = 0.6094265648090738$$

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3. Lower Bound

To find the lower bound, we must first prove a lemma which provides a frame for the lower bound formula, generalizing work in [3]. Then, we will apply the lemma to the Minkowski sausage. The result will be a computable equation which gives successively tighter lower bounds on the Hausdorff measure.

Lemma 3.1. *Suppose that a self-similar fractal E satisfies the open set condition with a common contraction ratio c and dimension $s > 0$. If there exists a value $d > 0$ such that at any level n of the construction there exists a set $K = \bigcup_{i=1}^{k} \Delta_i$ with $|K| \geq d$ that satisfies a_n, then $H^s(E) \geq a_n e^{-\frac{2s|E|}{d(1-c)}c^n}$.*

Proof. Let $K = \bigcup_{i=1}^{k} \Delta_i$, such that $\frac{|K|^s}{kc^{sn}} = a_n$, for some level n of the construction. Let $r = \frac{1}{c}$ for convenience. By the hypothesis, we can assume that $|K| \geq d$. Then there exists $\Delta^{n-1} \in E_{n-1}$ such that $\Delta_i \in \Delta_j^{n-1}$ for $j \in \{1, 2, ..., n-1\}$, where each Δ_j^{n-1} is different. By the scaling, it is clear that $k \leq mk_{n-1}$ and

$$\begin{aligned} \left| \bigcup_{j=1}^{k_{n-1}} \Delta_j^{n-1} \right| &\leq |K| + 2|E| \left(\frac{1}{r} \right)^{n-1} \\ &= |K| + 2|E| r \left(\frac{1}{r} \right)^{n} \end{aligned} \tag{1}$$

Thus,

$$\begin{aligned} \frac{|K|}{\left| \bigcup_{j=1}^{k_{n-1}} \Delta_j^{n-1} \right|} &\geq \frac{|K|}{|K| + 2|E| r \left(\frac{1}{r} \right)^{n}} \\ &= \frac{1}{1 + \frac{2|E|r}{|K|} \left(\frac{1}{r} \right)^{n}} \\ &\geq \frac{1}{1 + \frac{2|E|r}{d} \left(\frac{1}{r} \right)^{n}} \end{aligned}$$

Therefore since $k \leq mk_{n-1}$ and $m = r^s$,

$$\begin{aligned} \frac{|K|^s}{k\left(\frac{1}{m}\right)^n} &\geq \left(\frac{1}{1 + \frac{2|E|r}{d} \left(\frac{1}{r} \right)^{n}} \right)^s \frac{\left| \bigcup_{j=1}^{k_{n-1}} \Delta_j^{n-1} \right|^s}{k_{n-1}\left(\frac{1}{m}\right)^{n-1}} \\ \Rightarrow \frac{|K|^s}{k\left(\frac{1}{r}\right)^{sn}} &\geq \left(\frac{1}{1 + \frac{2|E|r}{d} \left(\frac{1}{r} \right)^{n}} \right)^s \frac{\left| \bigcup_{j=1}^{k_{n-1}} \Delta_j^{n-1} \right|^s}{k_{n-1}\left(\frac{1}{r}\right)^{s(n-1)}} \\ \Rightarrow a_n &\geq \left(\frac{1}{1 + \frac{2|E|r}{d} \left(\frac{1}{r} \right)^{n}} \right)^s a_{n-1} \end{aligned}$$

Since $|K| \geq d$ for all K that satisfy a_n at any n, we can unfix n and apply the above result $l \geq 1$ times for the formula

$$a_{n+l} \geq a_n \prod_{j=n+1}^{n+l} \left(\frac{1}{1 + \frac{2|E|r}{d} \left(\frac{1}{r} \right)^j} \right)^s$$

Take the logarithm on both sides and use the fact that $\ln(1+x) \leq x$ for all $x > 0$ to achieve the following:

$$\begin{aligned} \ln(a_{n+l}) &\geq \ln(a_n) - s \sum_{j=n+1}^{n+l} \ln \left(1 + \frac{2|E|r}{d} \left(\frac{1}{r} \right)^j \right) \\ &\geq \ln(a_n) - s \sum_{j=n+1}^{n+l} \left(\frac{2|E|r}{d} \left(\frac{1}{r} \right)^j \right) \end{aligned}$$

Let $l \to \infty$, giving us by proposition 1.2,

$$\begin{aligned} \ln(H^s(E)) &\geq \ln(a_n) - s \frac{\frac{2|E|r}{d} \left(\frac{1}{r} \right)^{n+1}}{1 - \frac{1}{r}} \\ &= \ln(a_n) - \frac{2s|E|r \left(\frac{1}{r} \right)^n}{d(r-1)} \\ &= \ln(a_n) - \frac{2s|E|}{d(1-c)} c^n \end{aligned}$$

And finally,

$$H^s(E) \geq a_n e^{-\frac{2s|E|}{d(1-c)} c^n}$$

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Remark. The larger d is, the tighter the bounds are. To this point, determining the maximum value for d gives us the upper limit to the effectiveness of this approximation.

Lemma 3.1 is of theoretical interest beyond the application to the Minkowski sausage; which fractals have such a value of d that can be easily found is an open question, and it is not obvious that $H^s(E) > 0$ must imply that $d > 0$, but we leave further discussion of this to the end of the paper. Before we determine d in the case of the Minkowski sausage, we can slightly improve the bounds formula itself for the specific application of this paper.

Corollary 3.1. *If $E = M$, i.e., the Minkowski sausage, then*

$$H^s(M) \geq a_n e^{-\frac{2s|E|}{d} c^n}$$

Proof. Notice that in the case of the Minkowski sausage, equation 1 can clearly be improved to

$$\begin{aligned}\left|\bigcup_{j=1}^{k_{n-1}} \Delta_j^{n-1}\right| &\leq |K| + \left(2|E| - 2|E|\left(\frac{1}{r}\right)\right)\left(\frac{1}{r}\right)^{n-1} \\ &= |K| + 2|E|\left(1 - \frac{1}{r}\right)\left(\frac{1}{r}\right)^{n-1} \\ &= |K| + 2|E|(r-1)\left(\frac{1}{r}\right)^{n}\end{aligned}$$

The rest of the proof follows exactly as in lemma 3.1, with an extra cancelation towards the end. ∎

Theorem 3.1. *The Hausdorff Measure of the Minkowski sausage satisfies*

$$H^s(M) \geq a_n e^{-36\sqrt{5}(\frac{1}{4})^n}$$

Proof. To prove the theorem, we use corollary 3.1 and find a suitable value of d for the Minkowski sausage (and as before, we calculate d based off of the constructor of the Minkowski sausage because the Minkowski sausage is a proper subset of the construction at any stage). Let $K = \bigcup_{i=1}^{k} \Delta_i$ be a collection of Δ_i at level n of the construction of M. Let $\Delta^0 = \{S_i(M) : S_i(M) \cap K \neq \emptyset\}, i = 1, 2, ..., 8$, that is, the eight immediate subsections of the Minkowski sausage. Then there are three cases, each dealing with how different elements of Δ^0 connect. Case 1 assumes that there exist elements of Δ^0 which are disconnected, case 2 handles if all the elements within Δ^0 are connected, and case 3 deals with the circumstance that Δ^0 contains only one element.

Case 1. There exist two elements of Δ^0 which are disconnected. Then the diameter of K is at least the minimum distance between two unconnected sets of Δ^0, which is the minimum distance between $S_2(M_0)$ and $S_4(M_0)$. Let d_1 be the distance between the lines $\overline{(\frac{1}{4}, 0)(\frac{1}{12}, \frac{1}{16})}$ and $\overline{(\frac{5}{12}, \frac{1}{12})(\frac{1}{2}, \frac{1}{4})}$, which is $\frac{\sqrt{5}}{20}$. See figure 4.

Case 2. Every element of Δ^0 is connected to every other element in the set. By the construction of M, Δ^0 can then only contain 2 elements, and by symmetry, we only need to consider two possibilities of connected sets, those being $\{\Delta_1, \Delta_2\}$ and $\{\Delta_4, \Delta_5\}$. We use a similar idea for both of them. Let $\Delta_1' = S_1 \circ S_8(M_0)$, $\Delta_2' = S_2 \circ S_1(M_0)$, $\Delta_4' = S_4 \circ S_8(M_0)$, and $\Delta_5' = S_5 \circ S_1(M_0)$. Suppose first that $\Delta^0 = \{\Delta_1, \Delta_2\}$:

a) If $K \cap (\Delta_1 - \Delta_1') \neq \emptyset$ or $K \cap (\Delta_2 - \Delta_2') \neq \emptyset$, then because removing Δ_1' or Δ_2' disconnects M, the diameter of K must be at least the minimum of the distances between $\Delta_1 - \Delta_1'$ and Δ_2, or between $\Delta_2 - \Delta_2'$ and Δ_1, respectively. By the figure 5, this distance is $\frac{\sqrt{5}}{60}$; let that be equal to d_2.

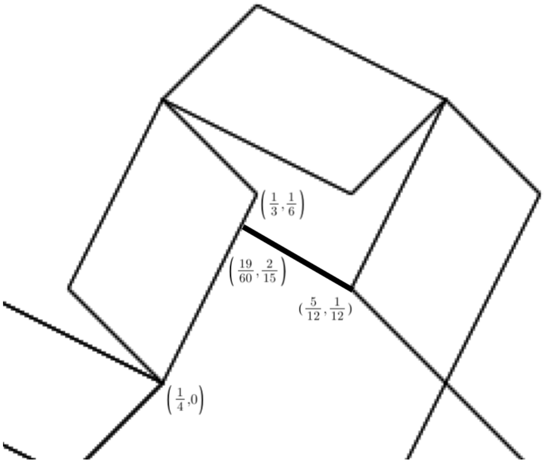

Figure 4. The shortest distance between two disconnected subsections.

b) if $K \in (\Delta_1' \cup \Delta_2')$, then by similarity, we can scale K by some integer 4^t about the point $\Delta_1 \cap \Delta_2$ such that this scaled K' satisfies case a), i.e., that K' no longer fits entirely within $(\Delta_1' \cup \Delta_2')$. Since $\frac{|K'|^s}{k'\left(\frac{1}{8}\right)^{n-t}} = \frac{|K|^s}{k\left(\frac{1}{8}\right)^n}$, K' also satisfies a_n, and the diameter is exactly similar to case a).

Supposing instead that $\Delta^0 = \{\Delta_4, \Delta_5\}$, it follows exactly as in the $\{\Delta_1, \Delta_2\}$ situation. The minimum distance here is the smallest distance between $\Delta_4 - \Delta_4'$ and Δ_5, or between $\Delta_5 - \Delta_5'$ and Δ_4, respectively. In figure 6, it is evident that this is equal to $\frac{\sqrt{5}}{48}$, which we will set as d_3.

Case 3. Δ^0 contains only one element. By symmetry, we can scale K up by a factor of some integer 4^t at some point p into a new K' such that $\Delta^0{\prime}$ now contains multiple elements and is covered under either case 1 or case 2 (where $\Delta^0{\prime}$ is defined analogously to Δ^0 but for K'). After defining k' again analogously to k, then under the scaling and as in case 2a, $\frac{|K'|^s}{k'\left(\frac{1}{8}\right)^{n-t}} = \frac{|K|^s}{k\left(\frac{1}{8}\right)^n}$, so K' satisfies a_n, and the diameter is therefore also similar to either case 1 or 2.

Since the three cases cover all possibilities for Δ^0, the diameter d of the set K must be at least $d \geq \min\{d_1, d_2, d_3\} = \min\left\{\frac{\sqrt{5}}{20}, \frac{\sqrt{5}}{48}, \frac{\sqrt{5}}{60}\right\} = \frac{\sqrt{5}}{60}$. By lemma 3.1, this gives a lower bound on the Minkowski sausage of

$$a_n e^{-\frac{2s|M|}{\sqrt{5}/60}\left(\frac{1}{c}\right)^n} = a_n e^{-36\sqrt{5}\left(\frac{1}{4}\right)^n}$$

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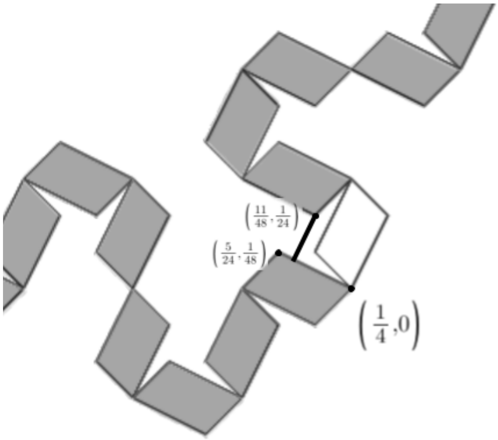

Figure 5. Δ_1 and Δ_2

Remark. By a computer calculation, $a_1 = 0.7217876047321581$. Thus, at $n = 1$, theorem 2 gives a lower bound of about 0.00000000131341377112, or $1.3 \cdot 10^{-9}$. However, due to the term $\frac{1}{4}^n$, the formula has an exponentially decreasing error and a quick convergence if values such as a_4 or a_5 can be found with accuracy.

4. Future research

Without a better algorithm to prune which branches to search, it is currently impractical in practice to find tight lower bounds using this method. In fact, to find even a_2 exactly, a naive algorithm must search more than eighteen quintillion different combinations. However, there is promise towards several areas for improvement.

If one can further restrict the cases where a given collection of Δ satisfy a_n, we can increase the value of d and therefore improve the lower bound formula. For example, in the Sierpinski gasket, there is evidence that any K satisfying a_n will always lie in all three large subtriangles, but the current lower bound featured in [3] features a lower bound as if K might be found in only two of the three subtriangles. Restricting the cases will also help in calculating a_n, since this would allow a computer to prune its searching program significantly.

Our first conjecture demonstrates the power of 2; if a way to find a_n for large values of n is discovered, then the lower bound converges almost immediately to a_n, and $H^s(M)$ is all but found.

Conjecture 1. $\theta_5 \approx a_5$, and so by theorem 2, $H^s(M) \geq 0.563353147953$.

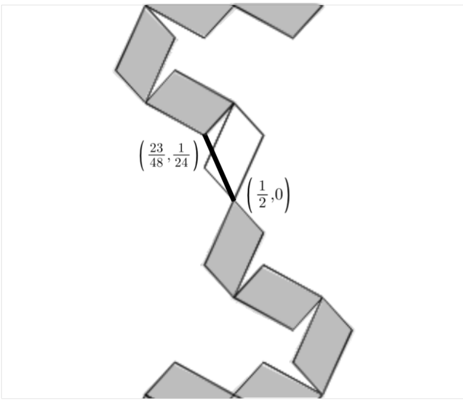

Figure 6. Δ_4 and Δ_5

lemma 3.1 provides an obvious way of extending the results of the paper. If d can be found reliably for many different fractals, then lower bounds formulas for all of those fractals are immediately found as well. However, we do not currently know if it is true that d even must be greater than zero for many fractals.

Conjecture 2. For a given fractal E, $d > 0$ if and only if $H^s(E) > 0$.

Conjecture 3. There exists a method to construct d for any fractal E if $d > 0$, using a technique similar to the scaling used in this paper.

These conjectures, if true, would strengthen both theorem 2 and lemma 3.1.

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References

[1] Michel Jébrak Dominique Bérubé. High precision boundary fractal analysis for shape characterization. *Computers and Geosciences*, 25(9):1059–1071, 1999.

[2] Kenneth J. Falconer. *The geometry of fractal sets*. Cambridge University Press, 1985.

[3] Baoguo Jia. Bounds of hausdorff measure of the sierpinski gasket. *Journal of Mathematical Analysis and Applications*, 330(2):1016–1024, June 2007.

[4] Baoguo Jia. Bounds of the hausdorff measure of the koch curve. *Applied Mathematics and Computation*, 190(1):559–565, July 2007.

[5] Alison Mcmillan. What is a straight line? does this matter in mechanical analysis? 11 2015.

[6] P. Móra. Estimate of the hausdorff measure of the sierpinski triangle. *Fractals*, 17:137–148, June 2009.

[7] Evgeni Ivanov Reiner Creutzburg, Adolf Mathias. Fast algorithm for computing the fractal dimension of binary images. *Physica A: Statistical Mechanics and its Applications*, 185(1-4):56–60, 1992.

[8] Minghua Wang. Some remarks on the hausdorff measure of the cantor set. *SHS Web of Conferences*, 24(01015), 2016.

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